QUADRATIC EQUATIONS

1. INTRODUCTION TO QUADRATIC EQUATIONS

1.1 What is a Quadratic Equation?

Definition: A quadratic equation in the variable x is an equation of the form:

$ax^2 + bx + c = 0$

Where:

- a, b, c are real numbers (constants)
- $a \neq 0$ (most important condition)
- x is the variable

Why $a \neq 0$? If $a = 0$, the equation becomes $bx + c = 0$, which is a linear equation, not quadratic.

Examples:

- $2x^2 + 3x - 5 = 0$ (Quadratic) ✓
- $x^2 - 4 = 0$ (Quadratic, $b = 0$) ✓
- $3x^2 + 7x = 0$ (Quadratic, $c = 0$) ✓
- $5x + 2 = 0$ (NOT quadratic, $a = 0$) ✗
- $x^3 + 2x^2 + 1 = 0$ (NOT quadratic, degree 3) ✗

1.2 Standard Form

Standard form: $ax^2 + bx + c = 0$

Where:

- a = Coefficient of x^2
- b = Coefficient of x
- c = Constant term

Example: Convert to standard form

Given: $2x^2 = 3x + 5$

Rearrange: $2x^2 - 3x - 5 = 0$

Here: $a = 2$, $b = -3$, $c = -5$

1.3 Roots (or Solutions) of Quadratic Equation

Definition: A value of x is called a root or solution of the quadratic equation if it satisfies the equation.

In other words: If α is a root of $ax^2 + bx + c = 0$, then:

$a\alpha^2 + b\alpha + c = 0$

Example: Is $x = 2$ a root of $x^2 - 5x + 6 = 0$?

LHS = $(2)^2 - 5(2) + 6$

$= 4 - 10 + 6$

$= 0$ = RHS

Yes, $x = 2$ is a root ✓

2. METHODS OF SOLVING QUADRATIC EQUATIONS

2.1 Method 1: Factorization

Steps:

Step 1: Write equation in standard form: $ax^2 + bx + c = 0$

Step 2: Factorize the quadratic expression

Step 3: Set each factor equal to zero

Step 4: Solve for x

When to use:

- When equation can be easily factorized
- When roots are rational numbers

Example 1: Solve $x^2 - 5x + 6 = 0$

Solution:

Step 1: Already in standard form

Step 2: Factorize

$x^2 - 5x + 6 = 0$

$x^2 - 2x - 3x + 6 = 0$

$x(x - 2) - 3(x - 2) = 0$

$(x - 2)(x - 3) = 0$

Step 3: Set factors to zero

$x - 2 = 0$ OR $x - 3 = 0$

Step 4: Solve

$x = 2$ OR $x = 3$

Roots: $x = 2$, $x = 3$

Example 2: Solve $2x^2 + 7x + 3 = 0$

Solution:

$2x^2 + 7x + 3 = 0$

$2x^2 + 6x + x + 3 = 0$

$2x(x + 3) + 1(x + 3) = 0$

$(2x + 1)(x + 3) = 0$

$2x + 1 = 0$ OR $x + 3 = 0$

$x = -1/2$ OR $x = -3$

Roots: $x = -1/2$, $x = -3$

Splitting the Middle Term:

For $ax^2 + bx + c = 0$:

1. Find two numbers whose product = $a \times c$
2. And whose sum = b
3. Split b using these two numbers
4. Factorize by grouping

Example: $6x^2 + 7x - 3 = 0$

$a \times c = 6 \times (-3) = -18$

Need two numbers: Product = -18, Sum = 7

Numbers: 9 and -2

$6x^2 + 9x - 2x - 3 = 0$

$3x(2x + 3) - 1(2x + 3) = 0$

$(3x - 1)(2x + 3) = 0$

$x = 1/3$ OR $x = -3/2$

2.2 Method 2: Completing the Square

Concept: Convert the quadratic expression into a perfect square form.

Perfect Square Forms:

$(x + a)^2 = x^2 + 2ax + a^2$

$(x - a)^2 = x^2 - 2ax + a^2$

Steps to Complete the Square:

Step 1: Write as: $x^2 + bx/a = -c/a$ (divide by 'a' if coefficient of $x^2 \neq 1$)

Step 2: Add $(b/2a)^2$ to both sides

Step 3: LHS becomes a perfect square

Step 4: Take square root of both sides

Step 5: Solve for x

Example 1: Solve $x^2 + 6x + 5 = 0$

Solution:

Step 1: $x^2 + 6x = -5$

Step 2: Add $(6/2)^2 = 9$ to both sides

$x^2 + 6x + 9 = -5 + 9$

$x^2 + 6x + 9 = 4$

Step 3: $(x + 3)^2 = 4$

Step 4: $x + 3 = \pm 2$

Step 5:

$x + 3 = 2$ OR $x + 3 = -2$

$x = -1$ OR $x = -5$

Roots: $x = -1$, $x = -5$

Example 2: Solve $2x^2 - 5x + 3 = 0$

Solution:

Divide by 2: $x^2 - 5x/2 + 3/2 = 0$

$x^2 - 5x/2 = -3/2$

Add $(5/4)^2 = 25/16$:

$x^2 - 5x/2 + 25/16 = -3/2 + 25/16$

$(x - 5/4)^2 = -24/16 + 25/16$

$(x - 5/4)^2 = 1/16$

$x - 5/4 = \pm 1/4$

$x = 5/4 + 1/4 = 6/4 = 3/2$

OR

$x = 5/4 - 1/4 = 4/4 = 1$

Roots: $x = 3/2$, $x = 1$

2.3 Method 3: Quadratic Formula

⭐ **MOST IMPORTANT METHOD** ⭐

For equation: $ax^2 + bx + c = 0$

Quadratic Formula:

$$x = [-b \pm \sqrt{(b^2 - 4ac)}] / 2a$$

Two Roots:

$x = [-b + \sqrt{(b^2 - 4ac)}] / 2a$ (Root 1)

$x = [-b - \sqrt{(b^2 - 4ac)}] / 2a$ (Root 2)

When to use:

- Always works for any quadratic equation
- Best when factorization is difficult
- When roots are irrational

Derivation from Completing the Square:

Starting with $ax^2 + bx + c = 0$:

$x^2 + (b/a)x + c/a = 0$

$x^2 + (b/a)x = -c/a$

Add $(b/2a)^2$:

$x^2 + (b/a)x + (b/2a)^2 = (b/2a)^2 - c/a$

$(x + b/2a)^2 = b^2/4a^2 - c/a$

$(x + b/2a)^2 = (b^2 - 4ac)/4a^2$

$x + b/2a = \pm\sqrt{(b^2 - 4ac)}/2a$

$x = -b/2a \pm \sqrt{(b^2 - 4ac)}/2a$

$x = [-b \pm \sqrt{(b^2 - 4ac)}] / 2a$

Example 1: Solve $x^2 - 7x + 10 = 0$

Solution:

$a = 1, b = -7, c = 10$

$x = [-(-7) \pm \sqrt{((-7)^2 - 4(1)(10))}] / 2(1)$

$x = [7 \pm \sqrt{(49 - 40)}] / 2$

$x = [7 \pm \sqrt{9}] / 2$

$x = [7 \pm 3] / 2$

$x = (7 + 3)/2 = 10/2 = 5$

OR

$x = (7 - 3)/2 = 4/2 = 2$

Roots: $x = 5, x = 2$

Example 2: Solve $2x^2 + 3x - 2 = 0$

Solution:

$a = 2, b = 3, c = -2$

$x = [-3 \pm \sqrt{(3^2 - 4(2)(-2))}] / 2(2)$

$x = [-3 \pm \sqrt{(9 + 16)}] / 4$

$x = [-3 \pm \sqrt{25}] / 4$

$x = [-3 \pm 5] / 4$

$x = (-3 + 5)/4 = 2/4 = 1/2$

OR

$x = (-3 - 5)/4 = -8/4 = -2$

Roots: $x = 1/2$, $x = -2$

3. NATURE OF ROOTS (DISCRIMINANT)

⭐ EXTREMELY IMPORTANT FOR EXAMS ⭐

3.1 Discriminant

Definition: For quadratic equation $ax^2 + bx + c = 0$, the expression $b^2 - 4ac$ is called the discriminant.

Symbol: D or Δ (Delta)

$D = b^2 - 4ac$

Why is it important? The discriminant tells us about the nature of roots WITHOUT actually solving the equation.

3.2 Nature of Roots Based on Discriminant

Case 1: $D > 0$ (Positive)

- Two distinct real roots
- Roots are different and real
- Example: $x^2 - 5x + 6 = 0$
- $D = (-5)^2 - 4(1)(6) = 25 - 24 = 1 > 0$Roots: $x = 2$, $x = 3$ (distinct real)

Case 2: $D = 0$ (Zero)

- Two equal real roots (repeated root)
- Roots are same and real
- Example: $x^2 - 4x + 4 = 0$
- $D = (-4)^2 - 4(1)(4) = 16 - 16 = 0$Roots: $x = 2$, $x = 2$ (equal real)

Case 3: $D < 0$ (Negative)

- No real roots
- Roots are imaginary/complex
- Example: $x^2 + x + 1 = 0$

- $D = (1)^2 - 4(1)(1) = 1 - 4 = -3 < 0$No real roots

Summary Table:

Discriminant	Nature of Roots	Example
$D > 0$	Two distinct real roots	$x^2 - 5x + 6 = 0$
$D = 0$	Two equal real roots	$x^2 - 4x + 4 = 0$
$D < 0$	No real roots	$x^2 + x + 1 = 0$

3.3 Examples on Discriminant

Example 1: Find the nature of roots of $2x^2 - 4x + 3 = 0$

Solution:

$a = 2, b = -4, c = 3$

$D = b^2 - 4ac$

$= (-4)^2 - 4(2)(3)$

$= 16 - 24$

$= -8 < 0$

Nature: No real roots (roots are imaginary)

Example 2: For what value of k does the equation $x^2 + 2x + k = 0$ have equal roots?

Solution:

For equal roots, $D = 0$

$D = b^2 - 4ac = 0$

$(2)^2 - 4(1)(k) = 0$

$4 - 4k = 0$

$4k = 4$

$k = 1$

Answer: k = 1

4. GRAPHICAL REPRESENTATION

4.1 Graph of Quadratic Equation

Equation: $y = ax^2 + bx + c$

Graph Shape: Parabola

Two Cases:

Case 1: $a > 0$ (Positive)

- Parabola opens upward (∪ shape)
- Minimum value exists

Case 2: $a < 0$ (Negative)

- Parabola opens downward (∩ shape)
- Maximum value exists

4.2 Roots and x-axis

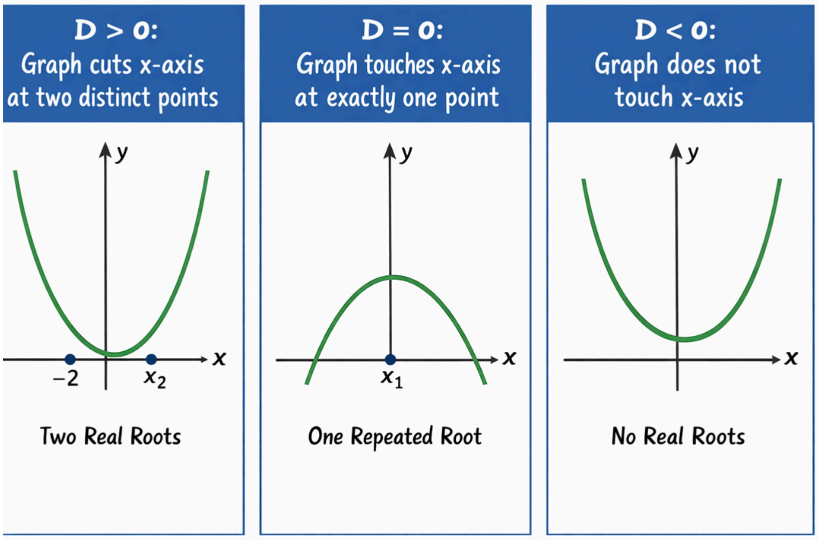
Roots of $ax^2 + bx + c = 0$ are the x-coordinates where the graph $y = ax^2 + bx + c$ intersects the x-axis.

Three Cases:

1. $D > 0$: Graph cuts x-axis at two distinct points

2. $D = 0$: Graph touches x-axis at exactly one point

3. $D < 0$: Graph does not touch x-axis



5. RELATIONSHIP BETWEEN ROOTS AND COEFFICIENTS

For equation: $ax^2 + bx + c = 0$ **Let roots be:** α and β

Important Relationships:

1. Sum of Roots:

$\alpha + \beta = -b/a$

2. Product of Roots:

$\alpha \times \beta = c/a$

3. Formation of Equation from Roots:

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

OR

x^2 - (Sum of roots)x + (Product of roots) = 0

Example 1: Find sum and product of roots of $2x^2 - 5x + 3 = 0$

Solution:

$a = 2, b = -5, c = 3$

Sum: $\alpha + \beta = -b/a = -(-5)/2 = 5/2$

Product: $\alpha\beta = c/a = 3/2$

Example 2: Form a quadratic equation whose roots are 3 and -2

Solution:

Sum = $3 + (-2) = 1$

Product = $3 \times (-2) = -6$

Equation: x^2 - (Sum)x + (Product) = 0

$x^2 - (1)x + (-6) = 0$

$x^2 - x - 6 = 0$

6. SOLVED EXAMPLES

Example 1: Solve by Factorization

Question: Solve $6x^2 - x - 2 = 0$

Solution:

$6x^2 - x - 2 = 0$

$6x^2 - 4x + 3x - 2 = 0$

$2x(3x - 2) + 1(3x - 2) = 0$

$(2x + 1)(3x - 2) = 0$

$2x + 1 = 0$ OR $3x - 2 = 0$

$x = -1/2$ OR $x = 2/3$

Answer: $x = -1/2, x = 2/3$

Example 2: Quadratic Formula

Question: Solve $x^2 - 4x + 1 = 0$

Solution:

$a = 1, b = -4, c = 1$

$x = [-(-4) \pm \sqrt{((-4)^2 - 4(1)(1))}] / 2(1)$

$x = [4 \pm \sqrt{(16 - 4)}] / 2$

$x = [4 \pm \sqrt{12}] / 2$

$x = [4 \pm 2\sqrt{3}] / 2$

$x = 2 \pm \sqrt{3}$

Answer: $x = 2 + \sqrt{3}, x = 2 - \sqrt{3}$

Example 3: Nature of Roots

Question: Find the discriminant and nature of roots: (i) $x^2 + 2x + 1 = 0$ (ii) $2x^2 + 5x + 5 = 0$

Solution:

(i) $x^2 + 2x + 1 = 0$

$D = (2)^2 - 4(1)(1) = 4 - 4 = 0$

Nature: Two equal real roots

(ii) $2x^2 + 5x + 5 = 0$

$D = (5)^2 - 4(2)(5) = 25 - 40 = -15 < 0$

Nature: No real roots

Example 4: Find Value of k

Question: If the equation $kx^2 + 6x + 1 = 0$ has real and equal roots, find k.

Solution:

For real and equal roots: $D = 0$

$D = b^2 - 4ac = 0$

$(6)^2 - 4(k)(1) = 0$

$36 - 4k = 0$

$k = 9$

Answer: $k = 9$

7. WORD PROBLEMS

Problem 1: Age Problem

Question: The product of two consecutive positive integers is 306. Find the integers.

Solution:

Let first integer = x Then second integer = x + 1

Given: $x(x + 1) = 306$

$x^2 + x = 306$

$x^2 + x - 306 = 0$

Using factorization:

$x^2 + 18x - 17x - 306 = 0$

$x(x + 18) - 17(x + 18) = 0$

$(x - 17)(x + 18) = 0$

$x = 17$ OR $x = -18$

Since integers are positive, $x = 17$

Answer: The integers are 17 and 18

Problem 2: Area Problem

Question: The length of a rectangle is 3 m more than its breadth. If the area is 180 m², find dimensions.

Solution:

Let breadth = x m Then length = (x + 3) m

Given: Area = 180

$x(x + 3) = 180$

$x^2 + 3x = 180$

$x^2 + 3x - 180 = 0$

Using formula:

$x = [-3 \pm \sqrt{(9 + 720)}] / 2$

$x = [-3 \pm \sqrt{729}] / 2$

$x = [-3 \pm 27] / 2$

$x = 24/2 = 12$ OR $x = -30/2 = -15$

Since breadth cannot be negative, $x = 12$

Answer: Breadth = 12 m, Length = 15 m

8. IMPORTANT FORMULAS

Quadratic Formula:

$x = [-b \pm \sqrt{(b^2 - 4ac)}] / 2a$

Discriminant:

$D = b^2 - 4ac$

Nature of Roots:

$D > 0$ → Two distinct real roots

$D = 0$ → Two equal real roots

$D < 0$ → No real roots

Sum and Product:

Sum of roots: $\alpha + \beta = -b/a$

Product of roots: $\alpha\beta = c/a$

Equation from Roots:

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

9. QUICK REVISION POINTS

Standard Form: **$ax^2 + bx + c = 0, a \neq 0$**

Three Methods:

1. **Factorization (when easy)**
2. **Completing the square (conceptual)**
3. **Quadratic formula (always works)**

Quadratic Formula:

$x = [-b \pm \sqrt{(b^2 - 4ac)}] / 2a$

Discriminant (D):

$D = b^2 - 4ac$

$D > 0$ → Two distinct real roots

$D = 0$ → Two equal real roots

$D < 0$ → No real roots

Sum and Product:

$\alpha + \beta = -b/a$

$\alpha\beta = c/a$

10. PRACTICE CHECKLIST

Before Exam:

□ Solve 30 equations by factorization

□ Solve 20 equations using quadratic formula

□ Practice discriminant for 25 equations

☐ Find nature of roots for 15 equations

☐ Solve 10 word problems

☐ Form 10 equations from given roots

☐ Find sum/product for 15 equations

☐ Practice completing the square (10 examples)

☐ Solve all NCERT exercises

☐ Attempt previous year questions

IMPORTANT: The quadratic formula and discriminant are the MOST IMPORTANT topics - they appear in almost every exam!

Formula to Remember:

$x = [-b \pm \sqrt{(b^2 - 4ac)}] / 2a$

$D = b^2 - 4ac$

All the Best for Board Exams! 🌟

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