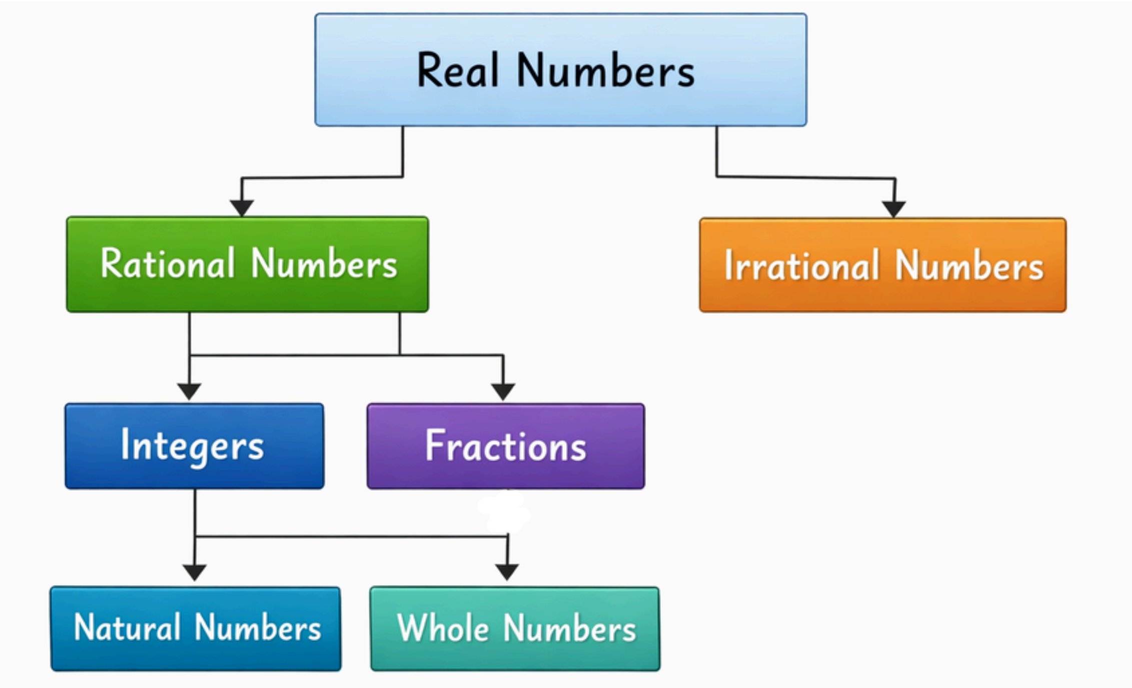
REAL NUMBERS

1. INTRODUCTION TO REAL NUMBERS

1.1 Number System Hierarchy



Natural Numbers (N): 1, 2, 3, 4, 5, ...

- Used for counting
- Smallest natural number = 1
- No largest natural number

Whole Numbers (W): 0, 1, 2, 3, 4, 5, ...

- Natural numbers + 0
- Smallest whole number = 0

Integers (Z): ..., -3, -2, -1, 0, 1, 2, 3, ...

- Positive integers, negative integers, and zero
- Z = {..., -3, -2, -1, 0, 1, 2, 3, ...}

Rational Numbers (Q): Numbers that can be expressed in the form p/q where p, q are integers and q ≠ 0

- Examples: 1/2, 3/4, -5/7, 0.5, 0.333..., 2 (can be written as 2/1)

- Decimal expansion: Terminating or Non-terminating recurring

Irrational Numbers (Q'): Numbers that CANNOT be expressed in the form p/q

- Examples: √2, √3, √5, π, e
- Decimal expansion: Non-terminating non-recurring
- Cannot be written as fractions

Real Numbers (R): All rational and irrational numbers together

- $R = Q \cup Q'$
- Every point on number line represents a real number

2. THE FUNDAMENTAL THEOREM OF ARITHMETIC

⭐ **MOST IMPORTANT TOPIC FOR 2025-26 BOARDS** ⭐

2.1 Statement

Every composite number can be expressed (factorized) as a product of primes, and this factorization is unique, apart from the order in which the prime factors occur.

In simple words:

- Every composite number = Product of prime numbers
- This representation is UNIQUE (only one way)
- Order doesn't matter ($2 \times 3 \times 5$ is same as $5 \times 2 \times 3$)

2.2 Important Definitions

Prime Number: A natural number greater than 1 that has exactly two factors - 1 and itself.

- Examples: 2, 3, 5, 7, 11, 13, 17, 19, 23, ...
- 2 is the only EVEN prime number
- 1 is NOT a prime number

Composite Number: A natural number greater than 1 that has more than two factors.

- Examples: 4, 6, 8, 9, 10, 12, 14, 15, ...

Co-prime Numbers (Relatively Prime): Two numbers are co-prime if their HCF is 1.

- Examples: (3, 5), (7, 9), (15, 16)
- Co-prime numbers may not be prime themselves

2.3 Prime Factorization

Method 1: Factor Tree Method

Example: Prime factorization of 60

```
    60
   / \
  2   30
     / \
    2   15
       / \
      3   5
```

$60 = 2 \times 2 \times 3 \times 5 = 2^2 \times 3 \times 5$

Method 2: Continuous Division Method

Example: Prime factorization of 120

```
2 | 120
2 | 60
2 | 30
3 | 15
5 | 5
  | 1
```

$120 = 2 \times 2 \times 2 \times 3 \times 5 = 2^3 \times 3 \times 5$

2.4 Standard Form

Any positive integer n can be written as:

$n = p_1^{a_1} \times p_2^{a_2} \times p_3^{a_3} \times \ldots \times p_k^{a_k}$

Where $p_1, p_2, p_3, \ldots, p_k$ are distinct primes and $a_1, a_2, a_3, \ldots, a_k$ are positive integers.

Examples:

- $24 = 2^3 \times 3$
- $90 = 2 \times 3^2 \times 5$
- $1000 = 2^3 \times 5^3$

3. HCF AND LCM USING PRIME FACTORIZATION

3.1 Finding HCF (Highest Common Factor)

Rule: **HCF = Product of smallest powers of common prime factors**

Steps:

1. **Find prime factorization of all numbers**
2. **Identify common prime factors**
3. **Take the smallest power of each common factor**
4. **Multiply them**

Example 1: Find HCF of 24 and 36

Solution:

$24 = 2^3 \times 3$

$36 = 2^2 \times 3^2$

Common prime factors: 2 and 3

- **For 2: Smallest power = $2^2 = 4$**
- **For 3: Smallest power = $3^1 = 3$**

HCF = $2^2 \times 3 = 4 \times 3 = 12$

Example 2: Find HCF of 96 and 404

Solution:

$96 = 2^5 \times 3$

$404 = 2^2 \times 101$

Common prime factor: 2 only

- **Smallest power of 2 = 2^2**

HCF = $2^2 = 4$

3.2 Finding LCM (Least Common Multiple)

Rule: **LCM = Product of greatest powers of all prime factors (common and uncommon)**

Steps:

1. **Find prime factorization of all numbers**
2. **Identify ALL prime factors (common and uncommon)**
3. **Take the greatest power of each factor**
4. **Multiply them**

Example 1: Find LCM of 24 and 36

Solution:

$24 = 2^3 \times 3$

$36 = 2^2 \times 3^2$

All prime factors: 2 and 3

- **For 2: Greatest power = $2^3 = 8$**
- **For 3: Greatest power = $3^2 = 9$**

LCM = $2^3 \times 3^2 = 8 \times 9 = 72$

Example 2: Find LCM of 12, 15, and 21

Solution:

$12 = 2^2 \times 3$

$15 = 3 \times 5$

$21 = 3 \times 7$

All prime factors: 2, 3, 5, 7

- **For 2: Greatest power = 2^2**
- **For 3: Greatest power = 3^1**
- **For 5: Greatest power = 5^1**
- **For 7: Greatest power = 7^1**

LCM = $2^2 \times 3 \times 5 \times 7 = 4 \times 3 \times 5 \times 7 = 420$

3.3 Important Relationship

For any two positive integers a and b:

$HCF(a, b) \times LCM(a, b) = a \times b$

This formula works ONLY for two numbers.

Example: If HCF(24, 36) = 12, find LCM(24, 36)

Solution:

$HCF \times LCM = 24 \times 36$

$12 \times LCM = 864$

LCM = 864/12 = 72

4. REVISITING RATIONAL NUMBERS

4.1 Definition

Rational Number: A number that can be expressed in the form p/q where:

- p and q are integers
- q ≠ 0

Examples:

- 3/4 (p = 3, q = 4)
- -5/7 (p = -5, q = 7)
- 0.5 = 1/2 (p = 1, q = 2)
- 2 = 2/1 (p = 2, q = 1)

4.2 Properties

1. Between any two rational numbers, there exist infinitely many rational numbers.

Example: Between 1/2 and 3/4:

- 5/8, 11/16, 9/16, 13/20, ... (infinite)

2. Rational numbers are DENSE on number line

3. Operations on rational numbers:

- Sum of two rationals = Rational
- Difference of two rationals = Rational
- Product of two rationals = Rational
- Quotient of two rationals = Rational (if denominator ≠ 0)

5. REVISITING IRRATIONAL NUMBERS

5.1 Definition

Irrational Number: A number that CANNOT be expressed in the form p/q (where p, q are integers and q ≠ 0)

Characteristics:

- Decimal expansion is non-terminating and non-recurring
- Cannot be written as a fraction

Examples:

- √2 = 1.41421356... (never ends, no pattern)
- √3 = 1.73205080...

- $\sqrt{5} = 2.23606797...$
- $\pi = 3.14159265...$
- $e = 2.71828182...$

5.2 Proving √2 is Irrational (Important Proof)

Given: √2 To Prove: √2 is irrational

Proof (By Contradiction):

Step 1: Assume that √2 is rational

Then √2 can be written as p/q where p and q are integers, $q \neq 0$, and p and q are co-prime (HCF = 1)

Step 2:

$\sqrt{2} = p/q$

Squaring both sides:

$2 = p^2/q^2$

$p^2 = 2q^2$ ... (1)

Step 3: From equation (1), p^2 is even (divisible by 2)

Step 4: If p^2 is even, then p must be even (if square is even, number is even)

So, p = 2m for some integer m

Step 5: Substitute p = 2m in equation (1):

$(2m)^2 = 2q^2$

$4m^2 = 2q^2$

$2m^2 = q^2$ ... (2)

Step 6: From equation (2), q^2 is even (divisible by 2)

Step 7: If q^2 is even, then q must be even

Step 8: We have proved:

- **p is even (divisible by 2)**
- **q is even (divisible by 2)**

This means p and q have a common factor 2

Step 9: But we assumed p and q are co-prime (HCF = 1)

This is a CONTRADICTION

Step 10: Our assumption was wrong

Therefore, $\sqrt{2}$ is irrational

Hence Proved

5.3 Similar Proofs

Using the same method, we can prove:

- $\sqrt{3}$ is irrational
- $\sqrt{5}$ is irrational
- $\sqrt{p}$ is irrational (where p is prime)

5.4 Operations on Irrational Numbers

1. Sum:

- Rational + Irrational = Irrational
  - Example: $2 + \sqrt{2}$ = Irrational
- Irrational + Irrational = May be rational or irrational
  - Example: $\sqrt{2} + \sqrt{2} = 2\sqrt{2}$ (Irrational)
  - Example: $\sqrt{2} + (-\sqrt{2}) = 0$ (Rational)

2. Difference:

- Rational - Irrational = Irrational
  - Example: $5 - \sqrt{3}$ = Irrational
- Irrational - Irrational = May be rational or irrational

3. Product:

- Rational × Irrational = Irrational (if rational ≠ 0)
  - Example: $2 \times \sqrt{3} = 2\sqrt{3}$ (Irrational)
- Irrational × Irrational = May be rational or irrational
  - Example: $\sqrt{2} \times \sqrt{2} = 2$ (Rational)
  - Example: $\sqrt{2} \times \sqrt{3} = \sqrt{6}$ (Irrational)

4. Division:

- Rational ÷ Irrational = Irrational (if rational ≠ 0)
  - Example: $4 \div \sqrt{2} = 4/\sqrt{2} = 2\sqrt{2}$ (Irrational)
- Irrational ÷ Irrational = May be rational or irrational

6. DECIMAL EXPANSIONS OF RATIONAL NUMBERS

6.1 Types of Decimal Expansions

1. Terminating Decimal:

- Decimal expansion ends after finite number of digits
- Examples: 0.5, 0.25, 0.125, 3.75

2. Non-terminating Recurring Decimal:

- Decimal expansion never ends but repeats a pattern
- Examples: 0.333... = $0.\overline{3}$, 0.142857142857... = $0.\overline{142857}$

3. Non-terminating Non-recurring Decimal:

- Decimal expansion never ends and never repeats
- These are irrational numbers
- Examples: √2 = 1.41421356..., π = 3.14159265...

6.2 Decimal Expansion of Rational Numbers

Key Theorem:

Let x = p/q be a rational number in lowest form (p and q are co-prime).

Then, the decimal expansion of x is:

1. TERMINATING if and only if the prime factorization of q has ONLY 2 or 5 or both as factors.

That is, q = 2^m × 5^n for some non-negative integers m and n.

2. NON-TERMINATING RECURRING if the prime factorization of q has factors OTHER THAN 2 and 5.

6.3 Checking Terminating or Non-terminating

Steps:

Step 1: Express the rational number in simplest form p/q (lowest terms)

Step 2: Find prime factorization of denominator q

Step 3: Check factors:

- If q = 2^m × 5^n only → Terminating
- If q has factors other than 2 and 5 → Non-terminating recurring

6.4 Examples

Example 1: Check if 7/8 is terminating or non-terminating

Solution:

7/8 is already in simplest form

Prime factorization of 8 = 2^3

Only factor is 2 (no 3, 7, or other primes)

Answer: Terminating

Actual value: 7/8 = 0.875 ✓

Example 2: Check if 13/125 is terminating or non-terminating

Solution:

13/125 is in simplest form

Prime factorization of 125 = 5^3

Only factor is 5

Answer: Terminating

Actual value: 13/125 = 0.104 ✓

Example 3: Check if 7/80 is terminating or non-terminating

Solution:

7/80 is in simplest form

Prime factorization of 80 = $2^4 \times 5$

Factors are only 2 and 5

Answer: Terminating

Actual value: 7/80 = 0.0875 ✓

Example 4: Check if 14/15 is terminating or non-terminating

Solution:

14/15 is in simplest form

Prime factorization of 15 = 3×5

Has factor 3 (other than 2 and 5)

Answer: Non-terminating recurring

Actual value: 14/15 = 0.933333... = 0.93 ✓

Example 5: Check if 17/6 is terminating or non-terminating

Solution:

17/6 is in simplest form

Prime factorization of 6 = 2 × 3

Has factor 3 (other than 2 and 5)

Answer: Non-terminating recurring

Actual value: 17/6 = 2.8333... = 2.83 ✓

Example 6: Check if 64/455 is terminating or non-terminating

Solution:

Step 1: Simplify if needed

64/455 is already in simplest form (HCF = 1)

Step 2: Prime factorization of 455

455 = 5 × 91 = 5 × 7 × 13

Step 3: Check factors

Denominator has factors 5, 7, 13

Has factors 7 and 13 (other than 2 and 5)

Answer: Non-terminating recurring

6.5 Important Points

1. Every rational number has either:

- Terminating decimal expansion, OR
- Non-terminating recurring decimal expansion

2. Irrational numbers have:

- Non-terminating non-recurring decimal expansion

3. To check terminating:

- Only look at denominator (after simplifying)
- Denominator should be of form $2^m \times 5^n$

4. If numerator and denominator share common factors:

- Always simplify first

- Then check denominator

7. IMPORTANT FORMULAS AND RESULTS

7.1 Key Formulas

1. HCF × LCM Formula (for two numbers):

HCF(a, b) × LCM(a, b) = a × b

2. Prime Factorization Standard Form:

$n = p_1^{a_1} \times p_2^{a_2} \times p_3^{a_3} \times \ldots \times p_k^{a_k}$

7.2 Important Results

1. HCF and LCM:

- HCF ≤ smaller of the two numbers
- LCM ≥ larger of the two numbers
- HCF divides both numbers
- Both numbers divide LCM

2. For co-prime numbers (HCF = 1):

- LCM(a, b) = a × b

3. Rational number p/q:

- Terminating if $q = 2^m \times 5^n$
- Non-terminating recurring otherwise

8. IMPORTANT THEOREMS

8.1 Fundamental Theorem of Arithmetic

Statement: Every composite number can be expressed as a product of primes uniquely (apart from order).

Application:

- Finding HCF using prime factorization
- Finding LCM using prime factorization
- Solving problems on divisibility

8.2 Theorem on Decimal Expansion

Statement: Let x = p/q be a rational number (in lowest form). The decimal expansion of x:

- Terminates if $q = 2^m \times 5^n$

- Non-terminating recurring otherwise

9. SOLVED EXAMPLES

Example 1: Prove that √3 is irrational

Solution (By Contradiction):

Assume √3 is rational = p/q (co-prime)

$\sqrt{3} = p/q$

$3 = p^2/q^2$

$p^2 = 3q^2$ ... (1)

From (1), p^2 is divisible by 3 → p is divisible by 3 → p = 3m for some integer m

Substitute in (1):

$(3m)^2 = 3q^2$

$9m^2 = 3q^2$

$3m^2 = q^2$

Therefore, q^2 is divisible by 3 → q is divisible by 3

So both p and q are divisible by 3 This contradicts our assumption that p and q are co-prime

Therefore, √3 is irrational

Example 2: Find HCF and LCM of 96 and 404 using prime factorization

Solution:

Prime Factorization:

$96 = 2^5 \times 3$

$404 = 2^2 \times 101$

HCF: Common prime factor: 2 only Smallest power: 2^2 HCF = 2^2 = 4

LCM: All prime factors: 2, 3, 101 Greatest powers: 2^5, 3^1, 101^1 LCM = $2^5 \times 3 \times 101 = 32 \times 3 \times 101 = 9696$

Verification: HCF × LCM = 4 × 9696 = 38784 a × b = 96 × 404 = 38784 ✓

Example 3: Check if 29/343 has terminating or non-terminating decimal expansion

Solution:

29/343 is in simplest form (HCF = 1)

Prime factorization of 343:

$343 = 7^3$

Denominator has factor 7 (other than 2 and 5)

Answer: Non-terminating recurring decimal expansion

Example 4: Check if 15/1600 has terminating decimal expansion

Solution:

Step 1: Simplify

15/1600 = 3/320 (dividing by 5)

Step 2: Prime factorization of 320

$320 = 2^6 \times 5$

Step 3: Check

Denominator = $2^6 \times 5$ (only 2 and 5)

Answer: Terminating decimal expansion

Example 5: Find the LCM and HCF of 6 and 20 by prime factorization. Verify that HCF × LCM = Product of numbers.

Solution:

Prime Factorization:

$6 = 2 \times 3$

$20 = 2^2 \times 5$

HCF: Common factor: 2 Smallest power: 2^1 HCF = 2

LCM: All factors: 2, 3, 5 Greatest powers: 2^2, 3^1, 5^1 LCM = $2^2 \times 3 \times 5 = 4 \times 3 \times 5 = 60$

Verification: HCF × LCM = 2 × 60 = 120 6 × 20 = 120 ✓

Verified

10. WORD PROBLEMS

Problem 1: Three bells toll at intervals of 9, 12, and 15 minutes. If they toll together at 8:00 AM, when will they toll together next?

Solution:

They will toll together after LCM(9, 12, 15) minutes

Prime Factorization:

$9 = 3^2$

$12 = 2^2 \times 3$

$15 = 3 \times 5$

$LCM = 2^2 \times 3^2 \times 5 = 4 \times 9 \times 5 = 180$ minutes = 3 hours

Answer: They will toll together at 11:00 AM (8:00 AM + 3 hours)

Problem 2: The length, breadth, and height of a room are 8 m 25 cm, 6 m 75 cm, and 4 m 50 cm. Find the length of the longest rod that can measure all three dimensions exactly.

Solution:

Convert to cm:

- Length = 825 cm
- Breadth = 675 cm
- Height = 450 cm

Longest rod = HCF(825, 675, 450)

Prime Factorization:

$825 = 3 \times 5^2 \times 11$

$675 = 3^3 \times 5^2$

$450 = 2 \times 3^2 \times 5^2$

$HCF = 3 \times 5^2 = 3 \times 25 = 75$ cm

Answer: The longest rod is 75 cm or 0.75 m

11. IMPORTANT POINTS TO REMEMBER

For HCF and LCM:

✓ HCF = Product of smallest powers of common prime factors

✓ LCM = Product of greatest powers of all prime factors

✓ HCF × LCM = Product of two numbers (only for 2 numbers)

✓ HCF ≤ Each number ≤ LCM

✓ If HCF = 1, numbers are co-prime

For Decimal Expansions:

✓ First simplify the fraction to lowest terms

✓ Check prime factorization of denominator only

✓ Terminating if denominator = $2^m \times 5^n$

✓ Non-terminating recurring if denominator has other prime factors

✓ All rational numbers → Terminating or Non-terminating recurring

✓ All irrational numbers → Non-terminating non-recurring

For Proofs:

✓ To prove $\sqrt{p}$ is irrational → Use contradiction method

✓ Assume $\sqrt{p} = a/b$ (co-prime)

✓ Show both a and b are divisible by p

✓ This contradicts co-prime assumption

✓ Therefore $\sqrt{p}$ is irrational

12. QUICK REVISION POINTS

Prime Numbers (first 15): 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47

Perfect Squares (first 15): 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225

Perfect Cubes (first 10): 1, 8, 27, 64, 125, 216, 343, 512, 729, 1000

Important Square Roots:

- $\sqrt{2} \approx 1.414$
- $\sqrt{3} \approx 1.732$
- $\sqrt{5} \approx 2.236$

Remember:

- HCF × LCM = a × b (for two numbers only)
- 2 is the only even prime
- 1 is neither prime nor composite
- Co-prime means HCF = 1
- Every even number > 2 is composite

13. PRACTICE CHECKLIST

Before Exam:

☐ Memorize first 20 prime numbers

☐ Practice prime factorization of 50 numbers

☐ Solve 10 HCF-LCM problems

☐ Check decimal expansion for 20 fractions

☐ Complete proof of √2, √3, √5 irrational (3 times each)

☐ Solve 5 word problems on HCF

☐ Solve 5 word problems on LCM

☐ Revise all theorems and statements

☐ Practice all NCERT exercise questions

☐ Solve previous year board questions

14. IMPORTANT DEFINITIONS

1. Prime Number: Natural number > 1 with exactly 2 factors (1 and itself)

2. Composite Number: Natural number > 1 with more than 2 factors

3. Co-prime Numbers: Two numbers whose HCF is 1

4. HCF: Highest Common Factor - largest number dividing all given numbers

5. LCM: Least Common Multiple - smallest number divisible by all given numbers

6. Rational Number: Number expressible as p/q ($q \neq 0$)

7. Irrational Number: Number NOT expressible as p/q

8. Real Number: All rational and irrational numbers together

9. Fundamental Theorem of Arithmetic: Every composite number has unique prime factorization

10. Terminating Decimal: Decimal that ends after finite digits

11. Non-terminating Recurring Decimal: Decimal that repeats a pattern endlessly

All the Best for Board Exams! 🌟

StudyByU