StudyByU

INTRODUCTION TO TRIGONOMETRY

1. INTRODUCTION TO TRIGONOMETRY

Trigonometry (from Greek: trigonon = triangle, metron = measure) is the branch of mathematics that deals with the relationship between the sides and angles of a triangle, particularly right-angled triangles.

Historical Note: Developed by ancient mathematicians including Hipparchus, Aryabhata, and Brahmagupta for astronomy and navigation.

Applications: Used in:

- Height and distance problems
- Navigation and surveying
- Architecture and engineering
- Astronomy and physics
- Computer graphics and game design

2. TRIGONOMETRIC RATIOS

Basic Concept

In a right-angled triangle, for any acute angle θ (theta), we define six trigonometric ratios based on the three sides:

Three Sides of a Right Triangle:

1. Hypotenuse - Side opposite to the right angle (longest side)
2. Perpendicular (Opposite side) - Side opposite to angle θ
3. Base (Adjacent side) - Side adjacent to angle θ

Important: The ratios depend on the angle, NOT on the size of the triangle.

The Six Trigonometric Ratios

For an acute angle θ in a right-angled triangle:

1. Sine (sin θ) = Perpendicular/Hypotenuse = P/H

2. Cosine (cos θ) = Base/Hypotenuse = B/H

3. Tangent (tan θ) = Perpendicular/Base = P/B

4. Cosecant (cosec θ) = Hypotenuse/Perpendicular = H/P = 1/sin θ

5. Secant (sec θ) = Hypotenuse/Base = H/B = 1/cos θ

6. Cotangent (cot θ) = Base/Perpendicular = B/P = 1/tan θ

Memory Tricks

Popular Mnemonic: "Some People Have Curly Brown Hair Through Proper Brushing"

- Some People Have → sin = P/H
- Curly Brown Hair → cos = B/H
- Through Proper Brushing → tan = P/B

Alternative Mnemonic: "Pandit Badri Prasad Har Har Bole, Bache Pad Pad Harre"**

- Pandit Badri Prasad → sin, cos, tan (numerators)
- Har Har Bole → denominators
- Bache Pad Pad Harre → cos, tan, sin (for base)

3. RECIPROCAL RELATIONSHIPS

Important Reciprocal Relations:

1. sin θ = 1/cosec θ OR cosec θ = 1/sin θ
2. cos θ = 1/sec θ OR sec θ = 1/cos θ
3. tan θ = 1/cot θ OR cot θ = 1/tan θ

Also:

1. tan θ = sin θ/cos θ
2. cot θ = cos θ/sin θ

4. TRIGONOMETRIC VALUES FOR STANDARD ANGLES

Standard Angles: 0°, 30°, 45°, 60°, 90°

Complete Table (MOST IMPORTANT - LEARN BY HEART):

Angle θ	0°	30°	45°	60°	90°
sin θ	0	1/2	1/√2	√3/2	1
cos θ	1	√3/2	1/√2	1/2	0
tan θ	0	1/√3	1	√3	Not defined
cosec θ	Not defined	2	√2	2/√3	1
sec θ	1	2/√3	√2	2	Not defined
cot θ	Not defined	√3	1	1/√3	0

Trick to Remember sin and cos Values:

For sin θ: √0/2, √1/2, √2/2, √3/2, √4/2 Simplified: 0, 1/2, 1/√2, √3/2, 1

For cos θ: Reverse of sin √4/2, √3/2, √2/2, √1/2, √0/2 Simplified: 1, √3/2, 1/√2, 1/2, 0

For tan θ: sin θ/cos θ 0, 1/√3, 1, √3, not defined

Pattern Recognition:

sin θ increases from 0° to 90°: 0 → 1 cos θ decreases from 0° to 90°: 1 → 0 tan θ increases from 0° to 90°: 0 → ∞

Key Observations:

- sin 0° = cos 90° = 0
- sin 30° = cos 60° = 1/2
- sin 45° = cos 45° = 1/√2
- sin 60° = cos 30° = √3/2
- sin 90° = cos 0° = 1

Complementary Angle Relationship: sin θ = cos (90° – θ) cos θ = sin (90° – θ)

5. DERIVATION OF STANDARD VALUES

For 45° (Using Isosceles Right Triangle)

In an isosceles right triangle with equal sides = 1:

- Two sides = 1, 1

- Hypotenuse = √2 (by Pythagoras)
- Each acute angle = 45°

Therefore:

- sin 45° = 1/√2
- cos 45° = 1/√2
- tan 45° = 1

For 30° and 60° (Using Equilateral Triangle)

Take an equilateral triangle with side = 2 units Draw perpendicular from one vertex to opposite side This creates two 30-60-90 triangles

In the right triangle:

- Hypotenuse = 2
- Base (for 30°) = 1
- Perpendicular = √3 (by Pythagoras)

For 30°:

- sin 30° = 1/2
- cos 30° = √3/2
- tan 30° = 1/√3

For 60°:

- sin 60° = √3/2
- cos 60° = 1/2
- tan 60° = √3

6. FUNDAMENTAL TRIGONOMETRIC IDENTITIES

Pythagorean Identities (MOST IMPORTANT)

Identity 1: $\sin^2\theta + \cos^2\theta = 1$

Derived Forms:

- $\sin^2\theta = 1 - \cos^2\theta$
- $\cos^2\theta = 1 - \sin^2\theta$

Identity 2: $1 + \tan^2\theta = \sec^2\theta$

Derived Forms:

- $\tan^2\theta = \sec^2\theta - 1$
- $\sec^2\theta - \tan^2\theta = 1$

Identity 3: $1 + \cot^2\theta = \csc^2\theta$

Derived Forms:

- $\cot^2\theta = \csc^2\theta - 1$
- $\csc^2\theta - \cot^2\theta = 1$

Proof of $\sin^2\theta + \cos^2\theta = 1$

In a right triangle ABC with ∠B = 90°:

- $\sin\theta = P/H$
- $\cos\theta = B/H$

By Pythagoras theorem: $P^2 + B^2 = H^2$

Dividing both sides by H^2:

- $P^2/H^2 + B^2/H^2 = H^2/H^2$
- $(P/H)^2 + (B/H)^2 = 1$
- $\sin^2\theta + \cos^2\theta = 1$ (Proved)

Proof of $1 + \tan^2\theta = \sec^2\theta$

We know: $\sin^2\theta + \cos^2\theta = 1$

Dividing both sides by $\cos^2\theta$:

- $\sin^2\theta/\cos^2\theta + \cos^2\theta/\cos^2\theta = 1/\cos^2\theta$
- $\tan^2\theta + 1 = \sec^2\theta$
- $1 + \tan^2\theta = \sec^2\theta$ (Proved)

Proof of $1 + \cot^2\theta = \csc^2\theta$

We know: $\sin^2\theta + \cos^2\theta = 1$

Dividing both sides by $\sin^2\theta$:

- $\sin^2\theta/\sin^2\theta + \cos^2\theta/\sin^2\theta = 1/\sin^2\theta$
- $1 + \cot^2\theta = \csc^2\theta$ (Proved)

7. COMPLEMENTARY ANGLES

Definition: Two angles are complementary if their sum is 90°.

Complementary Angle Relationships:

1. $\sin(90° - \theta) = \cos\theta$
2. $\cos(90° - \theta) = \sin\theta$
3. $\tan(90° - \theta) = \cot\theta$

4. cot (90° − θ) = tan θ
5. sec (90° − θ) = cosec θ
6. cosec (90° − θ) = sec θ

Examples:

- sin 30° = cos 60° = 1/2
- sin 40° = cos 50°
- tan 25° = cot 65°

8. IMPORTANT PROPERTIES AND OBSERVATIONS

Property 1: Range of Trigonometric Functions

- −1 ≤ sin θ ≤ 1
- −1 ≤ cos θ ≤ 1
- tan θ can take any value
- cosec θ ≥ 1 or cosec θ ≤ −1
- sec θ ≥ 1 or sec θ ≤ −1
- cot θ can take any value

Property 2: Values at Specific Angles

- sin 0° = 0, sin 90° = 1
- cos 0° = 1, cos 90° = 0
- tan 0° = 0, tan 90° = not defined

Property 3: Sign of Ratios For acute angles (0° to 90°), all trigonometric ratios are positive.

Property 4: Maximum and Minimum Values

- Maximum value of sin θ and cos θ = 1
- Minimum value of sin θ and cos θ = 0 (for acute angles)

9. WORKED EXAMPLES

Example 1: If sin θ = 3/5, find all other trigonometric ratios.

Solution: Given: sin θ = 3/5 = P/H So, P = 3, H = 5

Using Pythagoras: $B^2 = H^2 - P^2$

- $B^2 = 25 - 9 = 16$
- B = 4

Now:

- cos θ = B/H = 4/5
- tan θ = P/B = 3/4

- $\text{cosec}\ \theta = H/P = 5/3$
- $\sec\theta = H/B = 5/4$
- $\cot\theta = B/P = 4/3$

Example 2: If $\tan\theta = 1$, find $\sin\theta$ and $\cos\theta$.

Solution: $\tan\theta = 1$ means P = B

Let P = B = 1 Using Pythagoras: $H^2 = P^2 + B^2$

- $H^2 = 1 + 1 = 2$
- $H = \sqrt{2}$

Therefore:

- $\sin\theta = P/H = 1/\sqrt{2}$
- $\cos\theta = B/H = 1/\sqrt{2}$

This corresponds to $\theta = 45°$.

Example 3: Evaluate: sin 60° cos 30° + sin 30° cos 60°

Solution:

- $\sin 60° = \sqrt{3}/2$
- $\cos 30° = \sqrt{3}/2$
- $\sin 30° = 1/2$
- $\cos 60° = 1/2$

Substituting: $= (\sqrt{3}/2)(\sqrt{3}/2) + (1/2)(1/2) = 3/4 + 1/4 = 4/4 = 1$

Example 4: Prove that: $\tan^2\theta - \sin^2\theta = \tan^2\theta \times \sin^2\theta$

Solution: LHS $= \tan^2\theta - \sin^2\theta = \sin^2\theta/\cos^2\theta - \sin^2\theta = \sin^2\theta(1/\cos^2\theta - 1) = \sin^2\theta[(1 - \cos^2\theta)/\cos^2\theta]$ $= \sin^2\theta[\sin^2\theta/\cos^2\theta] = \sin^2\theta \times \tan^2\theta$ = RHS (Proved)

Example 5: If $\sec\theta = 13/12$, find other trigonometric ratios.

Solution: $\sec\theta = 13/12 = H/B$ So, H = 13, B = 12

Using Pythagoras: $P^2 = H^2 - B^2$

- $P^2 = 169 - 144 = 25$
- $P = 5$

Now:

- $\sin\theta = P/H = 5/13$
- $\cos\theta = B/H = 12/13$
- $\tan\theta = P/B = 5/12$

- cosec θ = H/P = 13/5
- cot θ = B/P = 12/5

Example 6: Verify: $\sin^2 30° + \cos^2 30° = 1$

Solution: LHS = $\sin^2 30° + \cos^2 30° = (1/2)^2 + (\sqrt{3}/2)^2 = 1/4 + 3/4 = 4/4 = 1$ = RHS (Verified)

Example 7: Express sin 75° + cos 75° in terms of complementary angles.

Solution: sin 75° = sin (90° − 15°) = cos 15° cos 75° = cos (90° − 15°) = sin 15°

Therefore: sin 75° + cos 75° = cos 15° + sin 15°

Example 8: If $\sin\theta + \sin^2\theta = 1$, find $\cos^2\theta + \cos^4\theta$.

Solution: Given: $\sin\theta + \sin^2\theta = 1$

- $\sin\theta = 1 - \sin^2\theta$
- $\sin\theta = \cos^2\theta$ ... (i)

To find: $\cos^2\theta + \cos^4\theta = \cos^2\theta + (\cos^2\theta)^2 = \cos^2\theta(1 + \cos^2\theta)$

From (i): $\cos^2\theta = \sin\theta$

So: $\cos^2\theta + \cos^4\theta = \sin\theta(1 + \sin\theta)$

From given: $\sin\theta + \sin^2\theta = 1$, so $1 + \sin\theta = 2 - \sin^2\theta$

But it's simpler to note: $\cos^2\theta + \cos^4\theta = \cos^2\theta(1 + \cos^2\theta) = \sin\theta(1 + \sin\theta) = \sin\theta + \sin^2\theta = 1$

10. TYPES OF PROBLEMS

Type 1: Finding Trigonometric Ratios

Given one ratio, find others using Pythagoras theorem.

Type 2: Evaluating Expressions

Substitute standard angle values and simplify.

Type 3: Proving Identities

Use fundamental identities to prove LHS = RHS.

Type 4: Simplification

Simplify expressions using identities and algebraic manipulation.

Type 5: Complementary Angles

Use complementary angle relationships.

Type 6: Finding Angle Values

Given a ratio value, identify the angle.

11. IMPORTANT FORMULAS AT A GLANCE

Basic Ratios:

- $\sin\theta = P/H$, $\cos\theta = B/H$, $\tan\theta = P/B$
- $\text{cosec}\,\theta = H/P$, $\sec\theta = H/B$, $\cot\theta = B/P$

Reciprocal Relations:

- $\sin\theta \times \text{cosec}\,\theta = 1$
- $\cos\theta \times \sec\theta = 1$
- $\tan\theta \times \cot\theta = 1$

Quotient Relations:

- $\tan\theta = \sin\theta/\cos\theta$
- $\cot\theta = \cos\theta/\sin\theta$

Pythagorean Identities:

- $\sin^2\theta + \cos^2\theta = 1$
- $1 + \tan^2\theta = \sec^2\theta$
- $1 + \cot^2\theta = \text{cosec}^2\theta$

Complementary Angles:

- $\sin(90° - \theta) = \cos\theta$
- $\cos(90° - \theta) = \sin\theta$
- $\tan(90° - \theta) = \cot\theta$

12. PROBLEM-SOLVING STRATEGIES

Strategy 1: When one ratio is given

- Identify P, B, H from the given ratio
- Use Pythagoras to find the missing side
- Calculate all other ratios

Strategy 2: For proving identities

- Start with the more complex side
- Use fundamental identities
- Simplify step by step
- Show LHS = RHS

Strategy 3: For evaluation

- Substitute standard angle values
- Simplify using arithmetic
- Rationalize if needed

Strategy 4: For simplification

- Look for identities that can be applied
- Convert all ratios to sin and cos if needed
- Factor common terms

13. IMPORTANT POINTS TO REMEMBER

✓ Learn the standard angle table by heart (0°, 30°, 45°, 60°, 90°)

✓ All three Pythagorean identities are equally important

✓ Complementary angle relationships are frequently tested

✓ Reciprocal relationships must be memorized

✓ Always rationalize the denominator in final answers

✓ For acute angles, all trigonometric ratios are positive

✓ tan 90° and cot 0° are not defined

✓ Maximum value of sin θ and cos θ is 1

✓ Use Pythagoras theorem when one ratio is given

✓ Identity proofs require step-by-step transformation

14. QUICK REVISION CHECKLIST

☐ Standard angle values (complete table)

☐ Six trigonometric ratios definitions

☐ Reciprocal relationships (3 pairs)

☐ Quotient relationships (tan, cot)

☐ Three Pythagorean identities

☐ Complementary angle formulas (6 formulas)

☐ Method to find all ratios from one ratio

☐ **Technique for proving identities**

☐ **Evaluation of expressions at standard angles**

☐ **Simplification using identities**

ALL THE BEST! 📐🔺✨