

# POLYNOMIALS

## 1. INTRODUCTION TO POLYNOMIALS

### 1.1 What is a Polynomial?

**Definition:** A polynomial in variable  $x$  is an algebraic expression of the form:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$$

Where:

- $a_n, a_{n-1}, \dots, a_2, a_1, a_0$  are real numbers (coefficients)
- $a_n \neq 0$
- $n$  is a non-negative integer
- $x$  is the variable

Examples:

- $p(x) = 2x^2 + 3x + 1$  (polynomial) ✓
- $p(x) = x^3 - 2x + 5$  (polynomial) ✓
- $p(x) = 1/x + 2$  (NOT a polynomial - negative power) ✗
- $p(x) = \sqrt{x} + 3$  (NOT a polynomial - fractional power) ✗

### 1.2 Degree of a Polynomial

**Definition:** The degree of a polynomial is the highest power of the variable in the polynomial.

Examples:

- $p(x) = 2x^3 + 3x^2 + 1 \rightarrow$  Degree = 3
- $p(x) = 5x^4 - 2x + 7 \rightarrow$  Degree = 4
- $p(x) = 7$  (constant)  $\rightarrow$  Degree = 0
- $p(x) = 0$  (zero polynomial)  $\rightarrow$  Degree = not defined

### 1.3 Types of Polynomials Based on Degree

Type	Degree	General Form	Example
Constant	0	$p(x) = a$	$p(x) = 5$
Linear	1	$p(x) = ax + b$	$p(x) = 2x + 3$
Quadratic	2	$p(x) = ax^2 + bx + c$	$p(x) = x^2 - 5x + 6$
Cubic	3	$p(x) = ax^3 + bx^2 + cx + d$	$p(x) = 2x^3 + 3x - 1$
Biquadratic	4	$p(x) = ax^4 + bx^3 + cx^2 + dx + e$	$p(x) = x^4 - 1$

Note: In Class 10, we mainly focus on quadratic polynomials.

## 1.4 Value of a Polynomial

If  $p(x) = 2x^2 + 3x + 1$ , find  $p(2)$

$$p(2) = 2(2)^2 + 3(2) + 1$$

$$= 2(4) + 6 + 1$$

$$= 8 + 6 + 1$$

$$= 15$$

## 2. ZEROES OF A POLYNOMIAL

### 2.1 Definition

Zero (or Root) of a Polynomial: A real number  $\alpha$  is called a zero of polynomial  $p(x)$  if  $p(\alpha) = 0$

In other words:

- When we substitute  $\alpha$  for  $x$  in  $p(x)$ , we get zero
- $\alpha$  is a solution of the equation  $p(x) = 0$

Example: Let  $p(x) = x^2 - 5x + 6$

Find if  $x = 2$  is a zero:

$$p(2) = (2)^2 - 5(2) + 6$$

$$= 4 - 10 + 6$$

$$= 0$$

Since  $p(2) = 0$ ,  $x = 2$  is a zero of  $p(x)$

## 2.2 Finding Zeroes Algebraically

For Linear Polynomial:  $p(x) = ax + b$

$$ax + b = 0$$

$$x = -b/a$$

Number of zeroes = 1

For Quadratic Polynomial:  $p(x) = ax^2 + bx + c$

Method 1: Factorization

Example:  $x^2 - 5x + 6 = 0$

$$(x - 2)(x - 3) = 0$$

$$x = 2 \text{ or } x = 3$$

Method 2: Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Number of zeroes = 2 (at most)

For Cubic Polynomial:  $p(x) = ax^3 + bx^2 + cx + d$  Number of zeroes = 3 (at most)

General Rule: A polynomial of degree  $n$  has at most  $n$  zeroes.

## 3. GEOMETRICAL MEANING OF ZEROES

### 3.1 Graphical Representation

**KEY CONCEPT:** The zeroes of a polynomial  $p(x)$  are the  $x$ -coordinates of the points where the graph of  $y = p(x)$  intersects the  $x$ -axis.

Why?

- At  $x$ -axis,  $y = 0$
- So,  $p(x) = 0$
- These are the zeroes of  $p(x)$

### 3.2 Linear Polynomial (Degree 1)

$$p(x) = ax + b$$

Graph: **Straight line**

Number of zeroes: **Exactly 1**

The line crosses x-axis at exactly one point.

Example:  $p(x) = 2x - 4$

$$2x - 4 = 0$$

$$x = 2$$

Graph crosses x-axis at  $x = 2$

### 3.3 Quadratic Polynomial (Degree 2)

$$p(x) = ax^2 + bx + c$$

Graph: **Parabola**

Three Cases:

**Case 1: Two Distinct Real Zeroes**

Parabola cuts x-axis at two distinct points

Example:  $p(x) = x^2 - 5x + 6 = (x-2)(x-3)$

- Zeroes:  $x = 2, x = 3$

**Case 2: Two Equal Real Zeroes (Repeated Root)**

Parabola touches x-axis at exactly one point

Example:  $p(x) = x^2 - 4x + 4 = (x-2)^2$

- Zero:  $x = 2$  (with multiplicity 2)

**Case 3: No Real Zeroes**

Parabola does not touch x-axis

Example:  $p(x) = x^2 + 1$

- No real zeroes (graph above x-axis)

### 3.4 Cubic Polynomial (Degree 3)

$$p(x) = ax^3 + bx^2 + cx + d$$

Graph: Cubic curve

Number of zeroes: 1, 2, or 3

Case 1: 3 zeroes Case 2: 1 zero

## 4. RELATIONSHIP BETWEEN ZEROES AND COEFFICIENTS

★ MOST IMPORTANT SECTION FOR BOARD EXAMS ★

### 4.1 For Quadratic Polynomial

Given: Quadratic polynomial  $p(x) = ax^2 + bx + c$  ( $a \neq 0$ ) Let:  $\alpha$  and  $\beta$  be the two zeroes

Important Relationships:

1. Sum of Zeroes:

$$\alpha + \beta = -b/a = -(\text{Coefficient of } x) / (\text{Coefficient of } x^2)$$

2. Product of Zeroes:

$$\alpha \times \beta = c/a = (\text{Constant term}) / (\text{Coefficient of } x^2)$$

3. Quadratic Polynomial from Zeroes: If  $\alpha$  and  $\beta$  are zeroes, then quadratic polynomial is:

$$p(x) = k[x^2 - (\alpha + \beta)x + \alpha\beta]$$

Where  $k$  is any non-zero constant

OR

$$p(x) = k[x^2 - (\text{Sum of zeroes})x + \text{Product of zeroes}]$$

For  $k = 1$ :

$$p(x) = x^2 - (\alpha + \beta)x + \alpha\beta$$

### 4.2 Examples

Example 1: Find the zeroes of  $x^2 - 3x + 2$  and verify the relationship

Solution:

$$x^2 - 3x + 2 = 0$$

$$(x - 1)(x - 2) = 0$$

$$x = 1 \text{ or } x = 2$$

Zeroes:  $\alpha = 1, \beta = 2$

**Verification:**

- $a = 1, b = -3, c = 2$
- Sum:  $\alpha + \beta = 1 + 2 = 3 = -(-3)/1 = 3 \checkmark$
- Product:  $\alpha \times \beta = 1 \times 2 = 2 = 2/1 = 2 \checkmark$

**Example 2: Find a quadratic polynomial whose zeroes are 3 and -2**

**Solution:**

$$\text{Sum of zeroes} = \alpha + \beta = 3 + (-2) = 1$$

$$\text{Product of zeroes} = \alpha \times \beta = 3 \times (-2) = -6$$

$$\text{Polynomial: } x^2 - (\text{Sum})x + (\text{Product})$$

$$= x^2 - (1)x + (-6)$$

$$= x^2 - x - 6$$

**Verification:**

$$(x - 3)(x + 2) = x^2 + 2x - 3x - 6$$

$$= x^2 - x - 6 \checkmark$$

**Example 3: If  $\alpha$  and  $\beta$  are zeroes of  $p(x) = 2x^2 - 5x + 3$ , find: (i)  $\alpha + \beta$  (ii)  $\alpha\beta$  (iii)  $\alpha^2 + \beta^2$  (iv)  $1/\alpha + 1/\beta$**

**Solution:**

$$\text{Here: } a = 2, b = -5, c = 3$$

$$\text{(i) } \alpha + \beta = -b/a = -(-5)/2 = 5/2$$

$$\text{(ii) } \alpha\beta = c/a = 3/2$$

$$\text{(iii) } \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

$$= (5/2)^2 - 2(3/2)$$

$$= 25/4 - 3$$

$$= 25/4 - 12/4$$

$$= 13/4$$

$$\text{(iv) } 1/\alpha + 1/\beta = (\alpha + \beta)/(\alpha\beta)$$

$$= (5/2)/(3/2)$$

$$= 5/3$$

### 4.3 For Cubic Polynomial

Given: Cubic polynomial  $p(x) = ax^3 + bx^2 + cx + d$  ( $a \neq 0$ ) Let:  $\alpha, \beta, \gamma$  be the three zeroes

Important Relationships:

1. Sum of Zeroes:

$$\alpha + \beta + \gamma = -b/a$$

2. Sum of Product of Zeroes Taken Two at a Time:

$$\alpha\beta + \beta\gamma + \gamma\alpha = c/a$$

3. Product of Zeroes:

$$\alpha\beta\gamma = -d/a$$

Note: This is for understanding only. Class 10 mainly focuses on quadratic polynomials.

## 5. DIVISION ALGORITHM FOR POLYNOMIALS

### 5.1 Division Algorithm Statement

If  $p(x)$  and  $g(x)$  are two polynomials with  $g(x) \neq 0$ ,

Then we can find polynomials  $q(x)$  and  $r(x)$  such that:

$$p(x) = g(x) \times q(x) + r(x)$$

Where:

- $p(x)$  = Dividend
- $g(x)$  = Divisor
- $q(x)$  = Quotient
- $r(x)$  = Remainder

Conditions:

1. Either  $r(x) = 0$
2. OR degree of  $r(x) <$  degree of  $g(x)$

This can be written as:

$$\text{Dividend} = \text{Divisor} \times \text{Quotient} + \text{Remainder}$$

### 5.2 Division Algorithm Steps

**Step 1: Arrange both polynomials in descending order of degree**

**Step 2: Divide the first term of dividend by first term of divisor to get first term of quotient**

**Step 3: Multiply divisor by this term and subtract from dividend**

**Step 4: The result becomes the new dividend**

**Step 5: Repeat until degree of remainder < degree of divisor**

### 5.3 Example of Polynomial Division

**Example: Divide  $3x^3 + x^2 + 2x + 5$  by  $1 + 2x + x^2$**

**Solution:**

**Step 1: Arrange in descending order**

**Dividend:  $3x^3 + x^2 + 2x + 5$**

**Divisor:  $x^2 + 2x + 1$**

**Step 2: Division process**

$$\begin{array}{r} 3x - 5 \\ \hline x^2 + 2x + 1 \mid 3x^3 + x^2 + 2x + 5 \\ 3x^3 + 6x^2 + 3x \quad (\text{Multiply: } x^2 + 2x + 1 \text{ by } 3x) \\ \hline -5x^2 - x + 5 \\ -5x^2 - 10x - 5 \quad (\text{Multiply: } x^2 + 2x + 1 \text{ by } -5) \\ \hline 9x + 10 \end{array}$$

**Result:**

**Quotient  $q(x) = 3x - 5$**

**Remainder  $r(x) = 9x + 10$**

**Verification:**

**$p(x) = g(x) \times q(x) + r(x)$**

**$3x^3 + x^2 + 2x + 5 = (x^2 + 2x + 1)(3x - 5) + (9x + 10)$**

**Expand RHS:**

$$= 3x^3 + 6x^2 + 3x - 5x^2 - 10x - 5 + 9x + 10$$

$$= 3x^3 + x^2 + 2x + 5 \checkmark$$

## 5.4 Special Cases

Case 1: When Remainder is 0

$$p(x) = g(x) \times q(x)$$

This means  $g(x)$  is a factor of  $p(x)$

Example:

$$x^2 - 5x + 6 \text{ divided by } x - 2$$

$$x^2 - 5x + 6 = (x - 2)(x - 3) + 0$$

Since remainder = 0,  $(x - 2)$  is a factor

Case 2: When  $g(x)$  is linear  $(x - a)$

If we divide  $p(x)$  by  $(x - a)$ , the remainder is  $p(a)$

This is called the Remainder Theorem

## 6. IMPORTANT FORMULAS AND IDENTITIES

### 6.1 For Quadratic Polynomial $ax^2 + bx + c$

1. Sum of zeroes:

$$\alpha + \beta = -b/a$$

2. Product of zeroes:

$$\alpha\beta = c/a$$

3. Polynomial from zeroes:

$$p(x) = x^2 - (\alpha + \beta)x + \alpha\beta$$

### 6.2 Useful Algebraic Identities

$$1. (\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$$

$$\text{Therefore: } \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

$$2. (\alpha - \beta)^2 = \alpha^2 + \beta^2 - 2\alpha\beta$$

Therefore:  $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

$$3. \alpha^3 + \beta^3 = (\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2)$$

$$= (\alpha + \beta)[(\alpha + \beta)^2 - 3\alpha\beta]$$

$$4. \alpha^3 - \beta^3 = (\alpha - \beta)(\alpha^2 + \alpha\beta + \beta^2)$$

$$5. 1/\alpha + 1/\beta = (\alpha + \beta)/(\alpha\beta)$$

$$6. \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

## 7. SOLVED EXAMPLES

### Example 1: Find zeroes and verify relationship

Question: Find the zeroes of  $6x^2 - 7x - 3$  and verify the relationship between zeroes and coefficients.

Solution:

Step 1: Arrange in standard form

$$6x^2 - 7x - 3 = 0$$

Step 2: Factorize

$$6x^2 - 9x + 2x - 3 = 0$$

$$3x(2x - 3) + 1(2x - 3) = 0$$

$$(3x + 1)(2x - 3) = 0$$

Step 3: Find zeroes

$$3x + 1 = 0 \text{ OR } 2x - 3 = 0$$

$$x = -1/3 \text{ OR } x = 3/2$$

Zeroes:  $\alpha = -1/3, \beta = 3/2$

Step 4: Verify

$$a = 6, b = -7, c = -3$$

$$\text{Sum: } \alpha + \beta = -1/3 + 3/2 = -2/6 + 9/6 = 7/6$$

$$-b/a = -(-7)/6 = 7/6 \checkmark$$

$$\text{Product: } \alpha \times \beta = (-1/3) \times (3/2) = -3/6 = -1/2$$

$$c/a = -3/6 = -1/2 \checkmark$$

### Example 2: Form polynomial from zeroes

Question: Find a quadratic polynomial whose zeroes are  $2 + \sqrt{3}$  and  $2 - \sqrt{3}$

Solution:

$$\alpha = 2 + \sqrt{3}$$

$$\beta = 2 - \sqrt{3}$$

$$\text{Sum: } \alpha + \beta = (2 + \sqrt{3}) + (2 - \sqrt{3}) = 4$$

$$\text{Product: } \alpha\beta = (2 + \sqrt{3})(2 - \sqrt{3})$$

$$= 2^2 - (\sqrt{3})^2 \quad [\text{Using } (a+b)(a-b) = a^2 - b^2]$$

$$= 4 - 3$$

$$= 1$$

$$\text{Polynomial: } x^2 - (\text{Sum})x + (\text{Product})$$

$$= x^2 - 4x + 1$$

### Example 3: Find value of k

Question: If one zero of the polynomial  $(k^2 + 4)x^2 + 13x + 4k$  is reciprocal of the other, find k.

Solution:

Let zeroes be  $\alpha$  and  $1/\alpha$  (reciprocal)

Product of zeroes:

$$\alpha \times (1/\alpha) = 1$$

$$\text{But } \alpha\beta = c/a$$

$$1 = 4k/(k^2 + 4)$$

$$k^2 + 4 = 4k$$

$$k^2 - 4k + 4 = 0$$

$$(k - 2)^2 = 0$$

$$k = 2$$

### Example 4: Division Algorithm

**Question:** Divide  $2x^2 + 3x + 1$  by  $x + 2$

**Solution:**

$$\begin{array}{r} 2x - 1 \\ \hline x + 2 \mid 2x^2 + 3x + 1 \\ \underline{2x^2 + 4x} \quad (\text{Multiply: } x + 2 \text{ by } 2x) \\ -x + 1 \\ \underline{-x - 2} \quad (\text{Multiply: } x + 2 \text{ by } -1) \\ 3 \end{array}$$

**Result:**

- Quotient =  $2x - 1$
- Remainder =  $3$

**Verification:**

$$\begin{aligned} 2x^2 + 3x + 1 &= (x + 2)(2x - 1) + 3 \\ &= 2x^2 - x + 4x - 2 + 3 \\ &= 2x^2 + 3x + 1 \checkmark \end{aligned}$$

## 8. IMPORTANT THEOREMS

### 8.1 Remainder Theorem

**Statement:** If a polynomial  $p(x)$  is divided by  $(x - a)$ , then the remainder is  $p(a)$ .

**Proof concept:**

**By division algorithm:**

$$p(x) = (x - a) \times q(x) + r$$

**Put  $x = a$ :**

$$p(a) = (a - a) \times q(a) + r$$

$$p(a) = 0 + r$$

$$p(a) = r$$

**Example:** Find remainder when  $x^3 - 3x^2 + 4x - 5$  is divided by  $(x - 2)$

Using Remainder Theorem:

$$\text{Remainder} = p(2)$$

$$= (2)^3 - 3(2)^2 + 4(2) - 5$$

$$= 8 - 12 + 8 - 5$$

$$= -1$$

## 8.2 Factor Theorem

**Statement:**  $(x - a)$  is a factor of  $p(x)$  if and only if  $p(a) = 0$

In other words:

- If  $p(a) = 0$ , then  $(x - a)$  is a factor
- If  $(x - a)$  is a factor, then  $p(a) = 0$

**Example:** Check if  $(x - 1)$  is a factor of  $x^3 - 2x^2 + x$

$$p(x) = x^3 - 2x^2 + x$$

$$p(1) = (1)^3 - 2(1)^2 + 1$$

$$= 1 - 2 + 1$$

$$= 0$$

Since  $p(1) = 0$ ,  $(x - 1)$  IS a factor

## 9. WORD PROBLEMS

### Problem 1: Age Problem

**Question:** The sum of two numbers is 27 and their product is 182. Find the numbers using polynomials.

**Solution:**

Let the numbers be  $\alpha$  and  $\beta$

Given:

- $\alpha + \beta = 27$
- $\alpha\beta = 182$

Form quadratic equation with these as zeroes:

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

$$x^2 - 27x + 182 = 0$$

Solve by factorization:

$$x^2 - 27x + 182 = 0$$

$$x^2 - 13x - 14x + 182 = 0$$

$$x(x - 13) - 14(x - 13) = 0$$

$$(x - 13)(x - 14) = 0$$

$$x = 13 \text{ or } x = 14$$

Numbers are 13 and 14

## 10. IMPORTANT POINTS TO REMEMBER

**Key Concepts:**

- ✓ Degree = Highest power of variable
- ✓ Zero of polynomial  $\rightarrow p(\alpha) = 0$
- ✓ Polynomial of degree  $n$  has at most  $n$  zeroes
- ✓ Graphically, zeroes =  $x$ -coordinates where graph cuts  $x$ -axis
- ✓ For quadratic: Sum =  $-b/a$ , Product =  $c/a$
- ✓ Polynomial from zeroes:  $x^2 - (\text{Sum})x + (\text{Product})$
- ✓ Division Algorithm:  $p(x) = g(x)q(x) + r(x)$
- ✓ Remainder Theorem: Remainder when divided by  $(x-a)$  is  $p(a)$
- ✓ Factor Theorem:  $(x-a)$  is factor  $\leftrightarrow p(a) = 0$

**For Solving Problems:**

- ✓ Always verify sum and product after finding zeroes
- ✓ Use factorization first, then quadratic formula if needed
- ✓ Check if remainder degree  $<$  divisor degree
- ✓ Use algebraic identities to simplify expressions
- ✓ Practice polynomial division thoroughly

# 11. QUICK REVISION POINTS

Quadratic Polynomial:  $ax^2 + bx + c$

If  $\alpha, \beta$  are zeroes:

1.  $\alpha + \beta = -b/a$

2.  $\alpha\beta = c/a$

3.  $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

4.  $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

5.  $1/\alpha + 1/\beta = (\alpha + \beta)/(\alpha\beta)$

Standard Quadratic from Zeroes:

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

Division Algorithm:

Dividend = Divisor  $\times$  Quotient + Remainder

$$p(x) = g(x) \times q(x) + r(x)$$

Important Identities:

$$(a + b)^2 = a^2 + b^2 + 2ab$$

$$(a - b)^2 = a^2 + b^2 - 2ab$$

$$a^2 - b^2 = (a + b)(a - b)$$

# 12. EXAM STRATEGIES

For 1-Mark Questions:

- Direct formula application
- Finding sum or product of zeroes
- Identifying degree
- True/False on properties

For 2-Mark Questions:

- Find zeroes by factorization
- Verify relationship
- Form polynomial from given zeroes
- Simple division problems

### For 3-Mark Questions:

- Find zeroes and verify completely
- Division algorithm problems
- Expression evaluation ( $\alpha^2 + \beta^2$ , etc.)
- Application of remainder theorem

### For 4-Mark Questions:

- Complete division with verification
- Word problems
- Multi-step problems
- Proof-based questions

## 13. PRACTICE CHECKLIST

### Before Exam:

- Find zeroes of 20 quadratic polynomials
- Verify sum and product for all
- Form 15 polynomials from given zeroes
- Solve 10 division algorithm problems
- Practice all algebraic identity applications
- Solve 5 word problems
- Memorize all formulas
- Practice graphical interpretation
- Solve all NCERT exercises (2.1, 2.2, 2.3)
- Attempt previous year questions

## 14. IMPORTANT GRAPH INTERPRETATIONS

### Number of Zeroes from Graph:

Linear (degree 1): Exactly 1 zero

Quadratic (degree 2): 0, 1, or 2 zeroes

Cubic (degree 3): 1, 2, or 3 zeroes

How to count: Count the number of times the graph intersects (or touches) the x-axis

All the Best for Board Exams! 🌟

